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ON THE CAPACITY OF GRAPHS

by

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To the memory of A. RÉNYI

The capacity of a graph $\mathcal{G}$ is the number of non-isomorphic (non-empty) subgraphs of $\mathcal{G}$, and is denoted by $\nu(n, \mathcal{G})$; here n indicates the number of vertices of $\mathcal{G}$ (or the order of $\mathcal{G}$, in notation $n = |\mathcal{G}|$). (Throughout this paper, subgraph will always mean induced subgraph.)

Put

$$\nu(n) = \max_{|\mathcal{G}|=n} \nu(n, \mathcal{G}) .$$

Obviously

$$n \leqq \nu(n) \leqq 2^n - 1 .$$

Goldberg conjectured that

$$\lim_{n\to\infty} \frac{\nu(n)}{2^n} = 0 .$$

In this paper we are going to disprove this conjecture, moreover, we establish the following

THEOREM 1.

$$\lim_{n\to\infty} \frac{\nu(n)}{2^n} = 1 .$$

Theorem 1 states that for large n there is a graph of whose subgraphs almost all are different (non-isomorphic).

Actually, we will show that almost all graphs have this property.

We say that almost all graphs have a given property, if the ratio of the number of graphs with n vertices not having this property to the total number of graphs with n vertices tends to 0 as $n \to \infty$. If this property does not depend on the labeling of the vertices of the graph (if there is any), then the above statement does not depend on the fact whether we consider the graphs labelled (and divide by $2^{\binom{n}{2}}$) or not-labelled (and divide by the number A_n of non

[1] This work was partially carried out while the second author spent an academic year at McGill University, Montreal.

isomorphic graphs with n vertices). This can be seen easily using the well-known asymptotic relation

$$A_n \sim \frac{2^{\binom{n}{2}}}{n!}.$$

We are going to prove the following sharpening of Theorem 1.

THEOREM 2. *For all graphs* $\mathcal{G}$

(1) $$\nu(n, \mathcal{G}) \leq 2^n - 2^{\left[\frac{n}{2}\right]-1},$$

and given $\varepsilon > 0$ *we have for almost all graphs*

(2) $$\nu(n, \mathcal{G}) > 2^n - 2^{\left(\frac{1}{2}+\varepsilon\right)n}.$$

Hence

$$\nu(n) = 2^n - 2^{\left(\frac{1}{2}+o(1)\right)n}.$$

Actually, we will prove a bit sharper estimation:

$$\nu(n) = 2^n - 2^{\frac{1}{2}n+O(\sqrt{n\log n})}.$$

However, $\nu(n, \mathcal{G})$ can be very small, since for the complete graph $\mathcal{G}_n$ $\nu(n, \mathcal{G}_n) = n$. Theorem 2 is in accordance with the fact that almost all graphs are not symmetric.

DEFINITION. The consistency set of a subset U of the vertices of a graph is the set V of vertices $v \notin U$, for which v is connected by either all vertices in U or none of them.

LEMMA. *In a graph of order n there always exist two vertices such that the consistency set of this pair of vertices contains at least* $\left[\frac{n}{2}\right] - 1$ *vertices.*

Lemma 1 obviously implies part (1) of Theorem 2, since putting one of these vertices and an arbitrary subset of their consistency set, and the other vertex and the same subset, the two subgraphs defined this way are isomorphic. Theorem 4 will state that in almost all graphs the above pairs of subgraphs are practically all pairs of isomorphic subgraphs.

PROOF of the lemma. Denote the valencies of the vertices by $v_1, v_2, \ldots, v_n$. Thus the sum of the numbers of vertices of consistency sets of all the $\binom{n}{2}$ pairs of vertices is obviously equal to

$$\sum_{i=1}^{n}\left[\binom{v_i}{2} + \binom{n-1-v_i}{2}\right] \geq n\,\frac{(n-1)(n-3)}{4},$$