Periodica Mathematica Hungarica Vol. 3 (1–2), (1973), pp. 13–17.

ON A "ROLLING" CHARACTERISTIC FUNCTION

by

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To the memory of A. RÉNYI

In 1970 ZOLOTARIOV posed the question, whether there exists a probability characteristic function, which has an infinite number of rolls around the O in a finite interval $[0, T)$, containing no roots of the characteristic function in question ($T > 0$). To give an exact formulation of the problem we introduce the following definition:

DEFINITION. We say that the continuous, complex-valued function $f(t)$, defined in the interval $[a, b]$, has at least M rolls around the O (M is a natural number), if $f(t) \neq 0$ ($t \in [a, b]$) and

$$\operatorname{arc} f(b) - \operatorname{arc} f(a) \geq M \cdot 2\pi. \tag{1}$$

We remark that $f(t) \neq 0$ implies that in $[a, b]$ it is possible to define the function $\operatorname{arc} f(t)$ continuously, and what is more, the difference on the left hand side of (1) does not depend on the way of definition of $\operatorname{arc} f(t)$.

The answer for the posed question is positive, at least partially; namely we shall prove the following

THEOREM. *There exists a characteristic function $\varphi(t)$ such that for a suitable $T > 0$*

$$\varphi(t) \neq 0 \qquad (t \in [0, T))$$

and it is possible to give the real numbers a_k, b_k ($k \geq 1$) so that

$$0 \leq a_1 < b_1 \leq a_2 < b_2 \leq \ldots < T,$$

and in each interval $[a_k, b_k]$ the function $\varphi(t)$ has at least one roll around the O.

In the example stated by the theorem the characteristic function $\varphi(t)$ has really an infinite number of rolls in the finite interval $[0, T)$ in that sense that there are an infinite number of disjoint subintervals, in each of them $\varphi(t)$ has at least one roll. But in the sense of our definition the number of rolls is not necessarily unbounded. In fact, we can not dominate the behaviour of $\varphi(t)$ in the intervals $[b_k, a_{k+1}]$, and it is possible that the values of the function arc $\varphi(t)$ oscillate in a bounded interval.

In consequence, from our example it follows that in general it is not possible to define the function arc $\varphi(t)$ continuously on the whole line, but the problem is open, whether there exists a characteristic function $\varphi(t)$, for which arc $\varphi(t)$ is unbounded in a finite interval.

Proof of the Theorem. In our construction we shall use the following lemma.

Lemma. *If $f(t)$ and $g(t)$ are continuous, complex-valued functions such that in the interval $[a, b]$ $f(t)$ has at least $M(\geq 2)$ rolls and for all $t \in [a, b]$*

$$|f(t)| > |g(t)|, \tag{2}$$

then in the same interval $f(t) + g(t)$ has at least $M - 1$ rolls.

Proof of the Lemma. From (2) it follows that for $t \in [a, b]$

$$f(t) + g(t) \neq 0$$

and that

$$|\operatorname{arc}(f(t) + g(t)) - \operatorname{arc} f(t)| \leq \frac{\pi}{2}.$$

Let us put here $t = a$ and $t = b$ and use (1). Then

$$\operatorname{arc}(f(b) + g(b)) - \operatorname{arc}(f(a) + g(a)) \geq (M - 1)2\pi + \pi.$$

The Lemma is proved.

The characteristic function stated in the theorem will be defined as follows:

$$\varphi(t) = \sum_{j=0}^{\infty} p_j (e^{in_j t} + e^{i(n_j + m_j)t}),$$

where

$$p_j \geq 0, \qquad 2\sum_{j=0}^{\infty} p_j = 1,$$

n_k is even, m_k is odd, and

$$n_j < n_j + m_j < n_{j+1}, \quad n_0 = 0. \tag{3}$$

In the proof we shall make further assumptions on the numbers p_k, m_k and n_k, and at the end of the proof we shall show that these numbers can be chosen in such a way that all the assumptions are satisfied.