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ON THE NUMBER OF CERTAIN HAMILTON CIRCUITS OF A COMPLETE GRAPH

by

G. BARÓTI (Budapest)

To the memory of A. RÉNYI

§ 0. Introduction

P. ERDŐS and A. RÉNYI have proposed the following problem: find the number $H_n(r)$ of Hamilton circuits having exactly r common edges with a fixed Hamilton circuit $\mathcal{H}$ of a complete linear graph G_n with n vertices.

Different recursion formulas for $H_n(r)$ are given by I. PALÁSTI (see [1], [2], [3], [4]) and by T. NEMETZ (see [6]). It is easy to see that

$$H_n(r) \leqq \frac{(n-1)!}{2} \qquad (0 \leqq r \leqq n),$$

since the number of all Hamilton circuits of order n is $\frac{(n-1)!}{2}$. I. PALÁSTI (see [5]) has shown that

$$H_n(0) \geqq (n-5)! \qquad (n \geqq 5)$$

and applied this result to the examination of Hamilton circuits of a random graph. Finally we mention that the following convergence relation is obtained by T. NEMETZ (see [6]):

$$\frac{H_n(0)}{(n-1)!} \to h(0) \qquad (n \to \infty)$$

where $0.038 < h(0) < 0.5$.

In this paper we are going to prove an explicit formula and an asymptotic expression for $H_n(r)$ by aid of the principle of inclusion and exclusion.

§ 1. The value of $H_n(r)$

THEOREM 1.

$$(1) \qquad H_n(r) = \sum_{s=0}^{n-r} (-1)^s \binom{r+s}{s} S_{r+s}^{(n)} \qquad (n \geqq 3,\ 0 \leqq r \leqq n)$$

where

$$S_k^{(n)} = \frac{(n-k-1)!}{2} \sum_{j=1}^{k} \left[\binom{k}{j} \binom{n-k-1}{j-1} + \binom{k-1}{j-1} \binom{n-k}{j} \right] 2^j \qquad (0 < k < n)$$

with

(2) $$S_0^{(n)} = \frac{(n-1)!}{2} \quad \textit{and} \quad S_n^{(n)} = 1 .$$

PROOF. Denoting the edges of $\mathcal{H}$ by $e_1, e_2, \ldots, e_n$ let A_i $(i = 1, 2, \ldots, n)$ be the property that the edge e_i appears in a Hamilton circuit of G_n and let $N(A_{i_1} A_{i_2} \ldots A_{i_k})$ denote the number of Hamilton circuits of G_n that have all the properties $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$. According to the principle of inclusion and exclusion in order to prove (1), it is enough to show that

(3) $$S_k^{(n)} \overset{\text{def}}{=} \sum N(A_{i_1} A_{i_2} \ldots A_{i_k})$$

(where the summation is over all k-combinations of the integers $1, 2, \ldots, n$) is given by (2) (see for example [7], Chapter 3).

We say that the edges $e_{i_1}, e_{i_2}, \ldots, e_{i_k}$ determine exactly j components if the graph obtained from $\mathcal{H}$ (which is a subgraph of G_n) by omitting the edges different from $e_{i_1}, e_{i_2}, \ldots, e_{i_k}$, consists of j components. Now consider a Hamilton circuit $\mathcal{H}^*$ of G_n which has all the properties $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ i.e. which contains the edges $e_{i_1}, e_{i_2}, \ldots, e_{i_k}$ and suppose that these edges determine exactly j components: $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_j$. Contract every component onto one point. By this contraction the graph G_n is mapped onto a complete graph G^*_{n-k} with $n-k$ vertices and $\mathcal{H}^*$ onto a Hamilton circuit $\mathcal{H}^*_{n-k}$ of G^*_{n-k}. Let $V_1, V_2, \ldots, V_j$ denote the vertices of G^*_{n-k} which are obtained by this contraction. Now consider an arbitrary Hamilton circuit $\mathcal{H}^*_{n-k}$ of G^*_{n-k} $(n-k \geq 3)$ and replace every vertex V_l by the component $\mathcal{C}_l$ (which is a path) $l = 1, 2, \ldots, j$ in the following way: omit V_l and the two edges $V_l V_l^{(1)}$, $V_l V_l^{(2)}$ which are adjacent to V_l in $\mathcal{H}^*_{n-k}$ and connect the vertices $V_l^{(1)}, V_l^{(2)}$ with the path $V_l^{(1)} C_l^{(1)} \ldots C_l^{(2)} V_l^{(2)}$ or $V_l^{(1)} C_l^{(2)} \ldots C_l^{(1)} V_l^{(2)}$, where $C_l^{(1)}, C_l^{(2)}$ are the endpoints of $\mathcal{C}_l$ (see Fig. 1). Since the number of possible replacements is 2^j, it can be obtained exactly 2^j Hamilton circuits $\mathcal{H}^*_{n;1}, \mathcal{H}^*_{n;2}, \ldots, \mathcal{H}^*_{n;2^j}$ from $\mathcal{H}^*_{n-k}$. Clearly each $\mathcal{H}^*_{n;l}$ $(l = 1, 2, \ldots, 2^j)$ is isomorphic to a Hamilton circuit $\mathcal{H}_{n;l}$ of G_n such that $\mathcal{H}_{n;l}$ has all the properties $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ and the edges $e_{i_1}, e_{i_2}, \ldots, e_{i_k}$ determine the components $\mathcal{C}_1, \mathcal{C}_2, \ldots, \mathcal{C}_j$. On the other

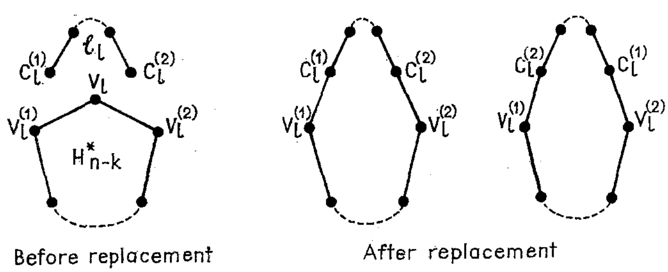

Fig. 1