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INCREASING PATHS IN EDGE ORDERED GRAPHS

by

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To the memory of A. RÉNYI

Given an undirected graph G having n vertices and q edges let the "ordering" N be a $1-1$ map between the edges of G and the positive integers $\leq q$. A path of length k is a sequence $(e_1, \ldots, e_k)$ of k distinct edges such that e_i and e_{i+1} have a common vertex. A path is simple if the only edges which have a common vertex are of the form e_i, e_{i+1} for some i. An increasing path is one in which $N(e_i) < N(e_j)$ whenever $i < j$.

The following questions have been raised by CHVÁTAL and KOMLÓS [1]: Suppose G is a complete graph K_n so that $q = \binom{n}{2}$. How long an increasing path must exist in G? How long a simple increasing path must exist? If we let $P(N, G)$ and $S(N, G)$ denote the lengths of the longest increasing and simple increasing paths, respectively, in G with the ordering N, then the preceding questions are concerned with

$$f(n) = \min_N P(N, K_n) \quad \text{and} \quad g(n) = \min_N S(N, K_n) .$$

In this note we give a complete answer to the first question and a partial answer to the second. In particular we show that for any edge ordered graph G having n vertices and q edges there is always an increasing path of length at least $2q/n$. From this it will follow that $f(3) = 3$, $f(5) = 5$, $f(n) = n - 1$ for $n \neq 3,5$. The length $g(n)$ of the longest simple increasing path in an edge ordered complete graph K_n has not been determined. We show that $g(n) \geq \frac{1}{2}(\sqrt{4n-3} - 1)$ but this is probably a weak bound, and we obtain a simple construction for which $g(n) < \frac{3n}{4}$. We conjecture that the correct bound is closer to the latter.

The results below are divided into four sections. The first two discuss the lower bounds on $P(N, G)$ and $S(N, K_n)$, respectively, and the latter two deal with the upper bound.

I. Lower bounds on P(N, G)

THEOREM 1. *The longest increasing path in any edge ordered graph G having n vertices and q edges has length at least $2q/n$.*

PROOF. Given an edge ordered graph G on n vertices $v_1, v_2, \ldots, v_n$ we define $p(v_k, G)$ to be the length of the longest increasing path ending at v_k. Suppose the edges of G' in order are $e_1, e_2, \ldots, e_{q'}$, and suppose that G'' has edges, in order, $e_1, \ldots, e_{q'}, e_{q'+1}$ with $e_{q'+1}$ joining v_r to v_s. Then

$$p(v_j, G'') \geqq p(v_j, G') \text{ for any } j,$$

$$p(v_r, G'') \geqq p(v_s, G') + 1$$

and

$$p(v_s, G'') \geqq p(v_r, G') + 1$$

since one can extend the longest increasing path ending at v_s in G' by the edge $e_{q'+1}$ arriving at a path of length $p(v_s, G') + 1$ ending at v_r, etc. Upon adding these relations we obtain

$$\sum_j p(v_j, G'') \geqq \Big(\sum_j p(v_j, G')\Big) + 2. \tag{1}$$

If we start from the empty graph and build up G edge by edge using this argument we obtain the result that if G has q edges then

$$\sum_{j=1}^{n} p(v_j, G) \geqq 2q$$

from which it follows that the average over j of $p(v_j, G)$ is at least $2q/n$. Thus, at least one v_j must have $p(v_j, G)$ as large as $2q/n$ and the theorem is proved.

The argument also indicates the type of ordering which will minimize the maximal $p(v_j, G)$. This will be one which, as far as possible, satisfies (1) with equality and for which all the $p(v_i, G)$ are as equal as possible.

For the complete graph K_n on n vertices, $q = \binom{n}{2}$ so that $\frac{2q}{n} = n - 1$. In Section III we show that for $n \neq 3,5$, one can find an ordering for which $p(v_j, K_n) = n - 1$ for all vertices v_j. This clearly can occur only if the inequalities (1) are always equalities in this ordering.

For $n = 3$, there is essentially only one possible ordering, and for this,

$$\max_j p(v_j, K_3) = 3 = f(3).$$

For $n = 5$, it is possible to exhaust the possible orderings; in each case there is an increasing path of length 5. This result can be verified in a less exhausting manner by an inspection of the sequences $(p(v_1, \bar{G}), p(v_2, \bar{G}), \ldots$ $\ldots, p(v_5, \bar{G}))$ with $\bar{G}$ denoting the graph which consists of the first five edges