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ON THE EIGENVALUES OF TREES

by

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To the memory of A. RÉNYI

Given a graph G (without loops and multiple edges) of n vertices labelled by $1, 2, \ldots, n$, we can form the adjacency matrix $A_G = (a_{ij})$ of G, defined by

$$a_{ij} = \begin{cases} 1 & \text{if the } i^{th} \text{ and } j^{th} \text{ vertices are joined by an edge,} \\ 0 & \text{otherwise.} \end{cases}$$

The adjacency matrix depends on the labelling of the vertices but its characteristic equation (and, consequently, its eigenvalues too) depend only on the graph G itself. As A_G is a symmetric matrix, these eigenvalues, called the eigenvalues of G, are real.

We denote by $f_G(\lambda)$ the characteristic polynomial $\det(\lambda I - A_G)$ of A_G and by $\Lambda(G)$ its largest root.

We shall begin with several general remarks on $f_G(\lambda)$ and $\Lambda(G)$, used in latter considerations. These propositions are special cases or easy consequences of general theorems on eigenvalues of non-negative matrices (see, e. g. [2] and [3]). Although they may be well-known for the reader, it may have some use to list them here.

Our main concern in this paper will be $f_G(\lambda)$ and $\Lambda(G)$ in the case when G is a tree (or more generally, a forest). We determine the maximal and minimal value of $\Lambda(G)$ among all trees of n vertices and give a method which enables us to determine the order of largest eigenvalues of two different trees in several cases.

NOTATIONS. $V(G)$ and $E(G)$ are the sets of vertices and edges of G, respectively. $G \cong G'$ means that G and G' are isomorphic. If G_1 and G_2 are arbitrary graphs then $G_1 + G_2$ is defined as follows: we consider a $G_1' \cong G_1$ and a $G_2' \cong G_2$ such that $V(G_1') \cap V(G_2') = \emptyset$ and let $V(G_1 + G_2) = V(G_1') + V(G_2')$, $E(G_1 + G_2) = E(G_1') + E(G_2')$. $G_1 + G_2$ is uniquely determined up to isomorphism. If $e \in E(G)$ and $x \in V(G)$ then $G - e$, $G - x$, $G - [e]$ denote the graphs arising from G by the removal of the edge e, of the vertex x and of the endpoints of e, respectively. If $e = (x,y)$ is a non-adjacent pair of vertices of G then $G \cup e$ denotes the graph obtained by adding the edge e to G. $G' \subseteq G$ means that $V(G') = V(G)$, $E(G') \subseteq E(G)$.

Proposition 1. *If G has at least one edge then $\Lambda(G) > 0$ and there is an eigenvector belonging to $\Lambda(G)$ with non-negative coordinates. If G is connected then $\Lambda(G)$ has multiplicity 1 and a positive eigenvector.*

Proposition 2. *If G' is a subgraph of G then $\Lambda(G') \leqq \Lambda(G)$.*

Proposition 3. *Let G_1, G_2 be two graphs on the same set of vertices. Then $\Lambda(G_1 \cup G_2) \leqq \Lambda(G_1) + \Lambda(G_2)$.*

Proposition 4. *Let $\varphi(G)$, $\Phi(G)$ denote the minimum and maximum valency of G. Then*

$$\max\left(\varphi(G),\ \sqrt{\Phi(G)}\right) \leqq \Lambda(G) \leqq \Phi(G).$$

Proposition 5. *A graph is bipartite iff its spectrum is symmetric to the origin.*

Proposition 6. *A connected graph is bipartite iff $-\Lambda(G)$ is an eigenvalue of it.*

Our investigations will be based on the following

Lemma 1. *If G is a forest and $e \in E(G)$ then*

$$f_G(\lambda) = f_{G-e}(\lambda) - f_{G-[e]}(\lambda).$$

Proof. As G is a forest we can label its vertices in such a way that e joins the points k and $k+1$ and there is no other edges between a point i $(1 \leqq i \leqq k)$ and a point j $(k+1 \leqq j \leqq n)$. Now the Laplace expansion of the determinant $\det(\lambda I - A_G)$ by its first k columns gives the equality of the lemma.

Theorem 1. *If G is a forest then*

$$f_G(\lambda) = \lambda^n - c_1\lambda^{n-2} + c_2\lambda^{n-4} \pm \ldots + (-1)^{\left[\frac{n}{2}\right]} c_{\left[\frac{n}{2}\right]} \lambda^{n-2\left[\frac{n}{2}\right]}$$

where c_k denotes the number of all k-element independent edge-systems in G.

Proof. We proceed by induction on the number of edges of G. For the empty graph of n vertices the theorem is obvious.

Let now e be an edge of G. Then $c_k = c_k' + c_k''$ where c_k' and c_k'' are the numbers of k-element independent edge-systems not containing e and containing e, respectively. Note that thus c_k' is the number of k-element independent edge-systems in $G-e$ while c_k'' is the number of $(k-1)$-element independent edge-systems in $G-[e]$. Now by induction $(-1)^k c_k'$ is the coefficient of λ^{n-2k} in $f_{G-e}(\lambda)$ and $(-1)^{k-1} c_k''$ is the coefficient of λ^{n-2k} in $f_{G-[e]}(\lambda)$. By Lemma 1 this proves the theorem.