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ON MÜNTZ–JACKSON'S THEOREM

by

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To the memory of A. RÉNYI

A classical theorem of Weierstrass states that to any function $f(x)$ defined and continuous in $[0,1]$ and any given $\varepsilon > 0$ there is a polynomial $\pi_n(x)$ such that

$$|f(x) - \pi_n(x)| < \varepsilon \tag{1}$$

for $0 \leq x \leq 1$.

There are different generalizations and improvements of this theorem. The first is the question to make "quantitative" the statement of the above theorem. More precisely if we restrict the degree of the approximating polynomials, how small can ε be chosen. This question has been cleared in 1912 by D. JACKSON [1]. His result is as follows.

Denote $\omega_f(\delta)$ the modulus of continuity of $f(x)$, that is

$$\omega_f(\delta) = \max_{\substack{x,h \\ |h| \leq \delta}} |f(x+h) - f(x)|,$$

then there is a polynomial of degree $\leq n$ such that

$$|f(x) - \pi_n(x)| \leq K\omega_f\left(\frac{1}{n}\right), \tag{2}$$

K being a numerical constant.

Weierstrass theorem can be formulated so that the linear-combinations of the powers of x:

$$1, x, x^2, \ldots \tag{3}$$

are dense in the space of functions continuous in $[0,1]$. Another question is whether the same is true of a subsequence of (3). S. BERNSTEIN [2] showed that a necessary and sufficient condition for the denseness of the linear-combinations of $\{x^{n_j}\}$ in $C[0,1]$ is $n_0 = 0$ and

$$\sum_{j=1}^{\infty} \frac{1}{n_j} = \infty. \tag{4}$$

As a generalization of Bernstein's result CH. MÜNTZ [3] showed that for *any* sequence $n_0 = 0 < n_1 < n_2, \ldots$ of real numbers the condition (4) is necessary and sufficient in order that the linear combinations of x^{n_j} $(j = 1,2,\ldots)$ should be dense in $C[0,1]$. (That is he dropped the condition that $n_0, n_1, \ldots$ should be a subsequence of the sequence of natural numbers). There are different proofs of Müntz' beautiful result. We mention that of O. SZÁSZ [4] based on a determinant formula of Cauchy (reproduced also in the book [5] of NATANSON) and the proof of PALEY—WIENER [6], based on their results on complex Fourier-transforms.

D. NEWMAN [7] combined the two generalizations of Weierstrass' theorem, asking about the accuracy of approximation to $f(x)$ by linear combinations of x^{n_j}. He proved the following theorem.

THEOREM. *If $f(x) \in L^2(0,1)$ and $n_{j+1} - n_j \geq 2$, then there is a $\pi_s^*(x)$ such that*

$$\| f(x) - \pi_s^*(x) \| \leq 3\omega_f^*(\varepsilon_s)$$

where $\pi_s^*(x)$ is a linear-combination of the x^{n_j}-s with $j \leq s$, $\|\cdot\|$ is the L^2-norm, $\omega_f^*(\delta)$ is the L^2-modulus of continuity of $f(x)$ and $\varepsilon_s = \prod_{0 \leq j \leq s} (n_j - 1/2)/(n_j + 3/2)$.

Further he showed that this result is essentially the best possible. The question for the accuracy of approximation in "uniform norm" remained open.

The purpose of the present paper is to give an upper estimate for the best approximation in the "uniform norm". We are not able to prove that our estimate is the best one; but in some special cases it is known that it is the best.

We prove the following two theorems:

THEOREM 1. *Let $\{n_j\}$ $(j = 1, 2, \ldots)$ be a sequence of real numbers satisfying*

$$n_0 = 0 \tag{5}$$

$$n_{j+1} - n_j \geq \delta > 0 \qquad (j = 0, 1, \ldots)\,. \tag{6}$$

Further we suppose that if $\delta \geq 1$, $n_1 = 2$. Then there is a $\Pi_s^*(x) = \sum_{j \leq s} a_j x^{n_j}$ such that

$$f(x) - \Pi_s^*(x) = O\left(\omega_f\left(\exp\left(-\frac{1}{M}\sum_{j\leq s} n_j^{-1}\right)\right)\right) \qquad (0 < x \leq 1) \tag{7}$$

uniformly in any closed subinterval of $(0, 1]$; *here M is a natural number satisfying $M \geq \delta^{-1}$.*

THEOREM 2. *Suppose that δ of* (6) *is equal to* 1. *Then* (7) *holds uniformly in the closed interval* $[0, 1]$.