Periodica Mathematica Hungarica Vol. 38 (1–2), (1999), pp. 15–17

ON A CONNECTION BETWEEN UNIT CIRCLES AND HOROCYCLES DETERMINED BY n POINTS

VOJTECH BÁLINT* (Žilina, Slovakia)
[Communicated by: Imre Bárány]

Abstract

In the paper we give the best possible estimate for the minimal number of unit circles determined by n points passing through an arbitrarily chosen point.

Given a set $\mathbf{A} \subset \mathbf{E}_2$ consisting of n points $P_1, P_2, \dots, P_n$ with $\operatorname{diam} \mathbf{A} < 2$, every pair of points of $\mathbf{A}$ determines exactly two unit circles. A unit circle is called of order m if and only if it contains exactly m points of $\mathbf{A}$; we denote the number of unit circles of order m by c_m. What is the maximum of c_m? A unit circle of order 2 is usually called an *ordinary* circle. Is $c_2 \geq 1$, i.e. does the set $\mathbf{A}$ determine at least one ordinary unit circle? What is the minimum of c_2? Let c denote the total number of unit circles determined by $\mathbf{A}$. What is the minimum of c? The best known estimate of c is due to Elekes [7]

$$c \geq n^{\frac{3}{2}}. \tag{1}$$

The following identities are obvious (see [1], [2]):

$$\sum_{j=2}^{n} j \cdot c_j = \sum_{j=2}^{c} j \cdot p_j, \tag{2}$$

where p_j is the number of points lying on exactly j unit circles, and

$$\sum_{j=2}^{n} \binom{j}{2} \cdot c_j = 2 \cdot \binom{n}{2}. \tag{3}$$

Let P_1, P_2, P_3 be the vertices of an inscribed acute triangle in a unit circle. Let P_4 be the intersection of the other three unit circles determined by pairs of P_1, P_2, P_3. Then the quadruple of points P_1, P_2, P_3, P_4 determines exactly four unit circles, all of order 3; this quadruple is called the *special 4-configuration*. Obviously, the special 4-configuration determines no ordinary unit circle. The question whether

Mathematics subject classification numbers, 52C10.
Key words and phrases. Horocycle, circle.

*Research supported by a Slovak National Grant VEGA 1/4332/97

0031-5303/99/$5.00

Akadémiai Kiadó, Budapest
Kluwer Academic Publishers, Dordrecht

$c_2 \geq 1$ for every point-set $\mathbf{A}$ such that $\operatorname{diam} \mathbf{A} \leq 2$ and $\mathbf{A}$ is different from the special 4-configuration was raised by A. Bezdek, F. Fodor and I. Talata at the 2nd Conference on *Convex and Discrete Geometry*, Bydgoszcz, 1998. They proved [6] that the answer is in the affirmative if $\operatorname{diam} \mathbf{A} \leq \sqrt{2}$.

THEOREM. *Let* $\mathbf{A}$ *be a set of* $n \geq 2$ *points in the Euclidean plane* $\mathbf{E}_2$ *such that* $\operatorname{diam} \mathbf{A} < 2$. *Then every point* $P_i \in \mathbf{A}$ *is incident with at least* $\frac{1+\sqrt{8n-7}}{2}$ *unit circles determined by the points of* $\mathbf{A}$.

PROOF. Let Ω be the circle of inversion with center P_n and radius 8. We call the circle $k_a(P_n)$, with center P_n and radius 2, the *antipodal circle of the point* P_n; apparently, all points of $\mathbf{A}$ lie in the interior of $k_a(P_n)$. The pair of points P_j, P_n, $j = 1, 2, \ldots, n$ determines exactly two unit circles, both touching the antipodal circle $k_a(P_n)$. By the inversion with respect to Ω, we get that the image of the antipodal circle $k_a(P_n)$ is a (concentric) circle $k'_a(P_n)$, and the image of the pair of unit circles determined by the points P_j, P_n is a pair of straight lines passing through P'_j, tangent to $k'_a(P_n)$. If u denotes the number of all such tangents to $k'_a(P_n)$, then $n - 1 \leq \binom{u}{2}$ because $P'_1, P'_2, \ldots, P'_{n-1}$ must lie in the intersection points of these tangent lines. Hence, $u \geq \frac{1+\sqrt{8n-7}}{2}$. □

REMARK 1. The estimate in our Theorem is sharp for $n = \frac{j^2+j+2}{2}$, $j = 1, 2, \ldots, n$, because we can obtain equality $n - 1 = \binom{u}{2}$ by a suitable choice of points.

REMARK 2. Consider the *Poincaré* model of the hyperbolic plane $\mathbf{H}_2$ in the Euclidean plane $\mathbf{E}_2$. The points of the hyperbolic plane are interpreted as inner points of the Euclidean upper half plane determined by the x-axis. *Horocycles* are either circles of the upper half-plane touching the x-axis or straight lines parallel to the x-axis. So every pair of points determines two horocycles.

E. Jucovič [9] asked the question, what the minimal number h of horocycles determined by n points is and J. Beck [5] proved

$$h > \beta \cdot n^2, \tag{4}$$

where β is an extremely small constant.

CONJECTURE 1.

$$h \approx \frac{1}{2} n^2.$$

Our Theorem holds also for horocycles (see [1], [2]). Because $\frac{1+\sqrt{8n-7}}{2} \approx \sqrt{2n} \ll n$ (compare e.g. with the number $\frac{33(n-1)}{247}$ of ordinary circles passing through each point; see [4]), it is possible that just this fact is a source of troubles by proving of the conjecture $c_2 \geq 1$, and also $h_2 \geq 1$ for the ordinary horocycles.