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CONTROLLING ORTHOGONAL SERIES WITH IRRATIONAL ROTATIONS

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Abstract

We compute covering numbers associated to the set of partial sums of orthogonal series by means of irrational rotations. The device is a new metric inequality linking the increment's norm of partial sums to the one of ergodic averages of rotations acting on a suitable L^2-element of the torus. This allows to compute the number of balls covering the whole set of partial sums, by means of rotations.

1. A metric inequality

Let $\mathbf{R}/\mathbf{Z} = [0,1[= \mathbf{T}$ be the torus with Lebesgue normalized measure m, and consider the characters $e_n(x) = \exp(2i\pi nx)$, $x \in \mathbf{T}$, $n \in \mathbf{Z}$. We use the notations $[x]$ and $\{x\}$, $x > 0$, where $[x]$ stands for the integral part of x and $\{x\} = x - [x]$ denotes its fractional part. Let $\theta \in \mathbf{T} \cap \mathbf{Q}^c$ be fixed, and $\tau x = x + \theta,\ mod(1)$, the rotation with angle θ. Define also the average kernels

$$\forall x \geq 1, \qquad V_x(\theta) = \frac{(e^{2i\pi x\theta} - 1)}{x(e^{2i\pi\theta} - 1)}.$$

Let $0 < \eta < 1$. By means of Weyl's criterion, we can build inductively two increasing sequences of positive integers (see section 2) $N_1 < N_2 < \cdots$ and $l_1 < l_2 < \cdots$, such that

$$(\Theta - 1) \qquad \forall j = 1, \cdots \forall N \leq N_j \qquad |V_N(l_j\theta)| > 1 - \frac{\eta}{2},$$

$$(\Theta - 2) \qquad \forall i = 1, \cdots \forall N \geq N_{i+1} \qquad |V_N(l_i\theta)| < \frac{\eta}{2}.$$

Now, let (r_k) be a sequence of positive integers; we denote $R_k = \sum_{j<k} r_j$ and put

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for any k

$$f_k = \frac{1}{\sqrt{r}_k} \sum_{R_k \le t < R_{k+1}} e_{l_t}$$

and

$$f = \sum_{k=0}^{\infty} c_k f_k$$

Let U be the unitary operator associated to τ by the relation $Uf = f \circ \tau$; we put $A_n^\theta = \frac{I+U+\dots+U^{n-1}}{n}$, $n \ge 1$. Since

$$A_N^\theta f - \sum_k c_k A_N^\theta f_k = \sum_k c_k \frac{1}{\sqrt{r}_k} \sum_{R_k \le t < R_{k+1}} e_{l_t} V_N(l_t\theta),$$

by orthonormality of the set of functions (f_k), we have

$$\left\|A_N^\theta f - A_M^\theta f\right\|^2 = \sum_k c_k^2 \frac{1}{r_k} \sum_{R_k \le t < R_{k+1}} \left|V_N(l_t\theta) - V_N(l_t\theta)\right|^2.$$

Now, evaluate $\left|V_N(l_t\theta) - V_M(l_t\theta)\right|^2$. Let t be such that $N \le N_t < N_{t+1} \le M$, and let k such that $R_k \le t < R_{k+1}$. By $(\Theta - 1)$, $(\Theta - 2)$

$$|V_N(l_t\theta)| > 1 - \frac{\eta}{2}, \qquad |V_M(l_t\theta)| \le \frac{\eta}{2}.$$

Then,

$$|V_N(l_t\theta) - V_M(l_t\theta)| \ge 1 - \eta.$$

Thus,

$$\left\|A_N^\theta f - A_M^\theta f\right\|^2 \ge (1-\eta)^2 \sum_k c_k^2 \frac{\sharp\{t \in [R_k, R_{k+1}[\, : \; N \le N_t < N_{t+1} \le M\}}{r_k}.$$

Let $h \ge k$, and assume that N and μ are such that

$$N \le N_{R_k} < N_{R_{h+1}} \le M.$$

Then, for any $t \in [R_u, R_{u+1}[$, with $u = k, \dots, h$, we have

$$N \le N_t < N_{t+1} \le M,$$

and by inequality (1)

$$\left\|A_N^\theta f - A_M^\theta f\right\|_{2,m}^2 \ge (1-\eta)^2 \sum_{u=k}^{h} c_u^2.$$

We thus have established the following proposition

PROPOSITION 1. *Let $\theta \in \mathbf{T}$ be irrational and $0 < \eta < 1$ be fixed. There exists two increasing sequences of positive integers $N_1 < N_2 < \cdots$ and $l_1 < l_2 < \cdots$ such*