Periodica Mathematica Hungarica Vol. 41 (1–2), (2000), pp. 121–147

ASYMPTOTIC INDEPENDENCE AND STRONG APPROXIMATION

A SURVEY

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Dedicated to Endre Csáki on the occasion of his 65th birthday

Abstract

Strong approximation results are discussed for the partial sum process of i.i.d. sequences of vectors having dependent components, where the components of the approximating process are independent. Many applications are considered for additive functionals in one and two dimensions.

1. Introduction

In this paper the results and developments of a strong approximation problem are discussed. Almost all the proofs are in papers published elsewhere. Our aim is to show the whole historic development of the topic and the broad range of applications which arised from it. Define the random walk $\{U_n\}_{n=1}^{\infty}$ on the line as $U_n = \sum_{k=1}^{n} Y_k$, where the random variables Y_i, $i = 1, 2, \ldots$ are i.i.d. taking integer values only. Define the local time of the walk by

$$\xi(x, n) = \#\{k;\ 0 \le k \le n,\ U_k = x\}. \tag{1.1}$$

Define

$$A_n = \sum_{i=1}^{n} f(U_i) = \sum_{x=-\infty}^{\infty} f(x)\xi(x, n) \tag{1.2}$$

where $f(.)$, $x \in \mathcal{Z}$ is a real valued function, satisfying

$$\sum_{x=-\infty}^{\infty} |f(x)| < \infty.$$

Mathematics subject classification numbers. 1991. Primary 60J55; Secondary 60F15, 60F17, 60J15.

Key words and phrases. Asymptotic independence, random walk, Brownian motion, additive functionals, invariance principle.

Research supported by a PSC CUNY Grant, No. 6-69413.

0031-5303/00/\$5.00
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Kluwer Academic Publishers, Dordrecht

A_n is a so-called additive functional, the behavior of which is a central topic of our investigations. The so called first-order results on A_n are establishing the following observation: the asymptotic behavior of A_n with appropriate normalization is the same as the behavior of the the local time. The interested reader should consult Kallianpur and Robbins [KR, 54], Darling and Kac [DK, 57], Skorohod and Slobodenyuk [SS, 70] and Borodin [B, 86b] to see the history of these first order limit results. However, here we focus our attention on the so-called second order limit theorems on A_n The history of this topic goes back to the landmark result of Dobrushin from 1955:

THEOREM A. (Dobrushin [D, 55]): *Assume that* $\mathbf{P}(Y_1 = +1) = \mathbf{P}(Y_1 = -1) = 1/2$. *If* $f(x)$ $x \in \mathcal{Z}$ *has finite support and* $\bar{f} = \sum_{x=-\infty}^{\infty} f(x) = 0$ *then*

$$\lim_{N\to\infty} \mathbf{P}\left(\frac{A_N}{dN^{1/4}} < x\right) = \mathbf{P}(U\sqrt{|V|} < x) \tag{1.3}$$

where U *and* V *are two independent standard normal variables, and*

$$d^2 = 4\sum_{k=-\infty}^{\infty} kf^2(k) + 8\sum\sum_{-\infty<i<j<\infty} if(i)f(j) - \sum_{k=-\infty}^{\infty} f^2(k).$$

This result has several generalizations. The corresponding functional version was given by Kasahara [K, 84]) and Borodin [B, 86b]. They proved that

$$\left(\sum_{i=1}^{[\lambda N]} f(U_i) - \bar{f}\xi(0,\lambda N)\right)(\lambda^\alpha L(\lambda))^{-1/2} \xrightarrow{w} dB(\ell_\alpha(N)) \qquad \text{as } \lambda\to\infty, \tag{1.4}$$

where $0 < \alpha < 1$ is a constant, $L(.)$ is a slowly varying function at infinity, $B(.)$ is a standard Wiener process, $\ell_\alpha(.)$ is the inverse process of the one-sided stable process with Laplace transform $\exp(-ts^\alpha)$, independent of $B(.)$ and $\xrightarrow{w}$ denotes weak convergence. Here α and $L(.)$ are determined by the asymptotic relation

$$\sum_{k=0}^{\infty} z^k P^{(k)} \sim (1-z)^{-\alpha} L\left(\frac{1}{1-z}\right) \qquad \text{as } z \to 1, \tag{1.5}$$

where $P^{(k)} = \mathbf{P}(U_k = 0)$. Kasahara [K, 84] proved (1.4) for general α, but $f(.)$ was assumed to have finite support, and Borodin [B, 86b] proved (1.4) for $\mathbf{E}|Y_i|^3 < \infty$, in which case $\alpha = 1/2$ $L(.) = \text{constant}$, while $f(x)$ was more general, than having finite support. The constant d was also explicitly given by Borodin [B, 86b]. Theorem A was extended to general random walk by Kesten [K, 62], Skorohod and Slobodenyuk [SS, 70] and Kasahara [K, 85].

The analogue of Theorem A for the Wiener process was proved by Skorohod and Slobodenyuk: