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# THE ACCURACY OF CAUCHY APPROXIMATION FOR THE WINDINGS OF PLANAR BROWNIAN MOTION

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*Dedicated to Professor Endre Csáki on the occasion of his $65^{th}$ birthday*

## 1. Introduction

Consider a 2–dimensional Brownian motion $(Z_t, t \geqslant 0)$ starting from $Z_0 = (1, 0)$ (for simplicity), and let $(\theta_t, t \geqslant 0)$ denote the continuous determination of the argument of $(Z_u, u \leqslant t)$ around $(0, 0)$ as $t$ evolves in $(0, \infty)$ such that $\theta_0 = 0$.

Spitzer [7] proved the celebrated result:

$$\lim_{t\to\infty} \mathsf{P}\left(\frac{\theta_t}{\log\sqrt{t}} < x\right) = G_0(x), \qquad x \in \mathbb{R}, \tag{1.1}$$

where $G_0$ denotes the standard Cauchy distribution function:

$$G_0(x) = \frac{1}{2} + \frac{1}{\pi}\arctan x, \qquad x \in \mathbb{R}.$$

We prove that for all $k \in \mathbb{N}$ the following expansion is valid:

$$\mathsf{P}\left(\frac{\theta_t}{\log\sqrt{t}} < x\right) = G_0(x) + \sum_{j=1}^{k-1} \frac{a_j}{(\log\sqrt{t})^j} G_j(x) + \mathsf{O}\left((\log t)^{-k}\right) \qquad \text{as } t \to \infty$$

uniformly in $x \in \mathbb{R}$, where

$$G_j(x) = \left.\frac{\mathrm{d}^j}{\mathrm{d}c^j}\right|_{c=1} G_0\left(\frac{x}{c}\right),$$

and the coefficients $a_j$ can be computed explicitly. We remark that the Fourier–Stieltjes transform of the function $G_j$ is

$$\int_{-\infty}^{\infty} \mathrm{e}^{i\lambda x}\,\mathrm{d}G_j(x) = (-|\lambda|)^j \mathrm{e}^{-|\lambda|}.$$

Our method of proof relies upon the skew–product representation of planar Brownian motion, and changes of probabilities between the laws of Bessel processes, as indicated in [8], [9], [10]; in particular, this paper is an improvement of [10], which discusses only the case $k = 2$.

The spirit of our expansion is close to the Edgeworth type expansions in the central limit theorems, that is, to the asymptotic expansions of the distribution

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function $F_n$ of $(X_1 + \cdots + X_n - a_n)/b_n$, where $(X_n)$ is a sequence of independent identically distributed random variables, and $(a_n)$, $(b_n)$ with $b_n > 0$ are sequences of constants such that $F_n$ tends to some stable law. If $\mathsf{E}|X_1|^2 < \infty$ then $F_n$ tends to the standard normal law as $n \to \infty$ where $a_n = n\mathsf{E}X_1$ and $b_n = \sqrt{n\mathsf{D}^2 X_1}$, and, if in addition, $\mathsf{E}|X_1|^{k+2} < \infty$ with some $k \in \mathbb{N}$ and Cramér's condition is fulfilled then

$$F_n(x) = \Phi(x) + \sum_{j=1}^{k-1} Q_j(x) n^{-j/2} + \mathsf{O}(n^{-k/2}) \qquad \text{as } n \to \infty$$

uniformly in $x \in \mathbb{R}$, where the functions $Q_j$ are linear combinations of derivatives of the standard normal distribution function $\Phi$ with some coefficients depending on the moments of $X_1$ up to the $(2+j)$–th order. See Petrov [5], Bhattacharya and Ranga Rao [1]. The question of asymptotic expansion in case of a non–normal stable law is less studied; see Christoph [2], Christoph and Wolf [3]. For example, if $X_1$ has a symmetric law such that for the logarithm of its Fourier transform the following expansion is valid:

$$\log \mathsf{E}\left[\mathrm{e}^{i\lambda X_1}\right] = -|\lambda| + \sum_{j=2}^{k} d_j |\lambda|^j + O(|\lambda|^{k+1}) \qquad \text{as } \lambda \to 0$$

for some $k \in \mathbb{N}$ then $F_n$ (with $a_n = 0$ and $b_n = n$) tends to the standard Cauchy law $G_0$ as $n \to \infty$, and, if in addition, Cramér's condition is fulfilled then

$$F_n(x) = G_0(x) + \sum_{j=1}^{k-1} \widetilde{Q}_j(x) n^{-j} + \mathsf{O}(n^{-k}) \qquad \text{as } n \to \infty$$

uniformly in $x \in \mathbb{R}$, where the functions $\widetilde{Q}_j$ are linear combinations of $G_\ell$, $\ell = j+1, \ldots, 2j$. (See Christoph [2, Theorem 2] or Christoph and Wolf [3, Theorem 4.12 and Corollary 4.20].)

## 2. Expansion of the Fourier transform

Spitzer's convergence result (1.1) is equivalent to

$$\lim_{t\to\infty} \mathsf{E}\left[\exp\left(\frac{i\lambda\theta_t}{\log\sqrt{t}}\right)\right] = \mathrm{e}^{-|\lambda|} \qquad \text{for all } \lambda \in \mathbb{R}. \tag{2.1}$$

The following statement gives an expansion of the Fourier transform of $\theta_t/\log\sqrt{t}$ in terms of the powers of $|\lambda|/\log\sqrt{t}$ as $t \to \infty$.

2.2 Proposition. *For every $k \in \mathbb{N}$, there exists a constant $C_k > 0$ such that for every $\lambda \in \mathbb{R}$ and for every $t > 1$ we have*

$$\left| \mathsf{E}\left[\exp\left(\frac{i\lambda\theta_t}{\log\sqrt{t}}\right)\right] - \mathrm{e}^{-|\lambda|} \sum_{j=0}^{k-1} a_j \left(-\frac{|\lambda|}{\log\sqrt{t}}\right)^j \right| \leqslant C_k \frac{|\lambda|\mathrm{e}^{-|\lambda|/2}}{(\log t)^k}$$