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PRE-SUPER BROWNIAN MOTION

Miklós Csörgő* (Ottawa) and Pál Révész** (Vienna)

Dedicated to Endre Csáki on the occasion of his 65th birthday

1. Introduction

The concept of branching Wiener process is due to Skorohod ([6]). This model can be described as follows.

Model 1

(i) a particle starts from the position $0 \in \mathbb{R}^d$ and executes a Wiener process $W(t) \in \mathbb{R}^d$,

(ii) arriving at time $t = 1$ to the new location $W(1)$, it dies,

(iii) at death it is replaced by Z offspring where

$$\mathbb{P}\{Z = \ell\} = p_\ell \quad (\ell = 0, 1, 2, \dots),$$

with

$$p_\ell \geq 0, \quad \sum_{\ell=0}^{\infty} p_\ell = 1, \quad \sum_{\ell=0}^{\infty} \ell\, p_\ell = m < \infty,$$

(iv) each offspring, starting from where its ancestor dies, executes a Wiener process (from its starting point) and repeats the above given steps, and so on. All Wiener processes and offspring-numbers are assumed to be independent of one-another.

A number of modified versions of this model are frequently used as well. Here we mention only two of them.

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Modification 1. (ii) can be restated by saying that the life-time of each particle is 1. Instead of this, it is assumed in many papers that the life-times are i.i.d. exponential r.v.'s.

Modification 2. According to (i) the motions of the particles are governed by independent Wiener processes. Instead of this, as it it is done in many papers, one may assume that the particles execute independent, simple, symmetric random walks in $\mathbb{Z}^d$

It is also natural to investigate such a model in the case when at time $t = 0$ we have a number of particles which execute independent branching Wiener processes according to Model 1. In fact we might consider two cases.

Case 1. At time $t = 0$ we have a Poisson random field $\pi = \{\mathbf{P}_1, \mathbf{P}_2, \dots\}$ $(\mathbf{P}_i \in \mathbb{R}^d)$ of parameter 1, i.e., in a Borel set $A \subset \mathbb{R}^d$ we have k particles with probability

$$\mathbb{P}\big\{\#\{i : \mathbf{P}_i \in A\} = k\big\} = \frac{|A|^k}{k!} e^{-|A|},$$

where $|A|$ is the Lebesgue measure of A. It is also assumed that the numbers of particles in disjoint Borel sets are independent r.v.'s. Each such particle executes an independent branching Wiener process starting from its position at time $t = 0$ according to Model 1.

Case 2. At time $t = 0$ we have N $(N = 1, 2, \dots)$ particles located at $0 \in \mathbb{R}^d$ which execute independent branching Wiener processes according to Model 1.

Case 2 has been one of the popular ones in these days, and it has been extensively investigated in the critical case, i.e., when $m = \sum_{\ell=0}^{\infty} \ell p_\ell = 1$, under the condition that N goes to infinity. The thus obtained limit process is one of the examples of how super-Brownian motion is obtained.

Our main goal is to give an elementary treatment of Case 2, first for N fixed, and then to investigate the asymptotic properties of the thus obtained results.

Naturally some of our asymptotic results may also follow from the general theory of super-Brownian motions (cf., e.g., [1], [2], [3]). However we believe that our simpler initial methods may nevertheless be of interest in studying and understanding the intrinsic nature of Case 2 in the critical case.

2. The general model and notations

At first we present a general model which contains the most important models investigated herewith.