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DIOPHANTINE EQUATIONS BETWEEN POLYNOMIALS OBEYING SECOND ORDER RECURRENCES

Peter Kirschenhofer (Leoben) and Oliver Pfeiffer (Leoben)

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Abstract

Let $\big(P_n(x)\big)_{n\geq 0}$ be a sequence of polynomials obeying a linear second order recurrence $P_{n+1}(x) = xP_n(x) + c_nP_{n-1}(x)$, $n \geq 0$, with rational parameters c_n. We give sufficient conditions depending on the parameters c_n under which the Diophantine equation $P_n(x) = P_m(y)$ has at most finitely many integer solutions.

1. Introduction

The polynomial Diophantine equation

$$P(x) = Q(y) \tag{1.1}$$

has been the subject of numerous investigations. Being a special case of the most general equation in two variables $F(x, y) = 0$ one can use the classical criterion of Siegel [11] to decide whether it has finitely or infinitely many integer solutions. Though this criterion completely solves the finiteness problem if the polynomials $P(x)$ and $Q(x)$ are explicitly given, it is in general not very helpful when they depend on unknown parameters. Davenport, Lewis and Schinzel [4] and Bilu and Tichy [2], besides many others, gave more explicit criterions for equation (1.1) in such cases, latter being the work which the present paper is based on.

We focus on more specific types of equation (1.1), namely those for which $P(x)$ and $Q(x)$ commonly belong to a sequence of monic polynomials defined by a linear second order recurrence of the form

$$\begin{gathered} P_{n+1}(x) = xP_n(x) + c_nP_{n-1}(x),\ n \geq 0, \\ P_{-1}(x) = 0,\ P_0(x) = 1, \end{gathered} \tag{1.2}$$

where $(c_n)_{n\geq 1}$ is a sequence of arbitrary rational numbers. Recurrences of this type and the finiteness problem for their associated Diophantine equations have

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been studied in the past for various choices of c_n, e.g. by Dujella and Tichy [5] for $c_n = \text{const.}$, by Kirschenhofer, Pethő and Tichy [7] for $c_n = n^2$, etc.

The aim of the present paper is to unify these results under more general, but primarily moderate conditions that are satisfied by a majority of "typical" polynomial recurrences, e.g. arising from combinatorial applications. A variety of enumeration problems lead to the definition of counting polynomials which obey recurrences of type (1.2). Kirschenhofer, Pethő and Tichy [7] and Bilu, Stoll and Tichy [3], for example, considered the problem of determining the number of integral points in higher dimensional octahedrons and obtained a recurrence of exactly the pattern of (1.2). A combinatorial problem leading to a related recurrence has been studied by the authors in a previous paper [8].

A different type of linear second order recurrences

$$G_{n+1}(x) = p(x)G_n(x) + q(x)G_{n-1}(x),\ n \geq 0,$$

where the coefficients $p(x)$ and $q(x)$ depend on x, but are constant with respect to n, has been studied by Fuchs, Pethő and Tichy [6]. In this paper, the authors considered Diophantine equations of the form $G_n(x) = G_m\big(P(x)\big)$ and gave explicit bounds on the number of pairs $(n, m) \in \mathbb{Z}^2$ for which this decomposition is possible.

2. Preliminaries and main result

Recently, Bilu and Tichy [2] were able to prove a criterion that describes the conditions under which a Diophantine equation of type (1.1) has infinitely many so-called *solutions with bounded denominator*. A Diophantine equation is said to have infinitely many solutions with bounded denominator if there exists a positive integer Δ such that $P(x) = Q(y)$ has infinitely many rational solutions (x, y) with $\Delta x, \Delta y \in \mathbb{Z}$.

Another term introduced in [2] which is of importance for our concerns are the so-called *standard pairs*. Throughout the following listing of the five types of standard pairs let $p(x)$ denote a polynomial with rational coefficients, $\alpha \neq 0$, β and δ rational numbers and s and t non-negative integers. Let the *Dickson polynomials* $D_n(x, \delta)$ be defined by

$$D_n\left(z + \frac{\delta}{z}, \delta\right) = z^n + \left(\frac{\delta}{z}\right)^n.$$

Explicitly, these polynomials read

$$D_n(x, \delta) = \sum_{k \geq 0} d_{n,k} x^{n-2k} \quad \text{with } d_{n,k} = \frac{n}{n-k}\binom{n-k}{k}(-\delta)^k,$$

from which we obtain

$$d_{n,0} = 1, \quad d_{n,1} = -n\delta, \quad d_{n,2} = \frac{n(n-3)}{2}\delta^2, \quad \ldots$$