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VARIOUS NORMS ON INDEFINITE INNER PRODUCT SPACES

by

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1. Statement of the problem and main result

Let $\mathfrak{H}$ be a vector space over the complex numbers, and $(.\,,.)$ a hermitian form (to be called the *inner product)* on $\mathfrak{H}$. Suppose that $\mathfrak{H}$ can be written as a direct sum,

$$\mathfrak{H} = \mathfrak{H}^+ \dot{+} \mathfrak{H}^-, \tag{1.1}$$

where a) $\mathfrak{H}^+$ is *orthogonal* to $\mathfrak{H}^-$, i.e., $(x^+, x^-) = 0$ for every pair $x^+ \in \mathfrak{H}^+$, $x^- \in \mathfrak{H}^-$; b) $\mathfrak{H}^+$ and $\mathfrak{H}^-$ are Hilbert spaces relative to the hermitian forms

$$[x^+, y^+]_{\mathfrak{H}^+} = (x^+, y^+) \qquad (x^+, y^+ \in \mathfrak{H}^+) \tag{1.2}$$

and

$$[x^-, y^-]_{\mathfrak{H}^-} = -(x^-, y^-) \qquad (x^-, y^- \in \mathfrak{H}^-), \tag{1.3}$$

respectively.

If the above conditions are satisfied, which we shall constantly assume, we say $\mathfrak{H}$ is a *Krein space* and (1.1) is a *fundamental decomposition* of $\mathfrak{H}$ (cf. [1]).

Set

$$[x^+ + x^-, y^+ + y^-] = (x^+, y^+) - (x^-, y^-) \qquad (x^+, y^+ \in \mathfrak{H}^+;\ x^-, y^- \in \mathfrak{H}^-) \tag{1.4}$$

and

$$p(x) = [x, x]^{1/2} \qquad (x \in \mathfrak{H}). \tag{1.5}$$

The norm p will be called the *fundamental norm* corresponding to the fundamental decomposition (1.1) (in [1] we used the terms "J-norm" and „natural norm").

If $\mathfrak{H}^+ \neq 0$, $\mathfrak{H}^- \neq 0$, then $\mathfrak{H}$ has several fundamental decompositions (this follows easily by reduction to the 2-dimensional case) and, consequently, several fundamental norms. The set of all fundamental norms on $\mathfrak{H}$ will be denoted by Φ.

It can be proved (see e.g. [1; Corollary IV. 6.3]) that each $p \in \Phi$ induces the same topology $\tau = \tau(\mathfrak{H})$. Also the cardinal numbers

$$\varkappa^+(\mathfrak{H}) = \dim \mathfrak{H}^+, \qquad \varkappa^-(\mathfrak{H}) = \dim \mathfrak{H}^- \tag{1.6}$$

do not depend on how the fundamental decomposition (1.1) is selected (cf. [1; Corollary IV. 7.4]).

Let T be a τ-continuous linear operator on $\mathfrak{H}$. Each fundamental norm $p \in \Phi$ defines a norm $p(T)$ of the operator T according to the formula

$$p(T) = \sup_{\substack{x \in \mathfrak{H} \\ x \neq 0}} \frac{p(Tx)}{p(x)} \qquad (p \in \Phi). \tag{1.7}$$

We put the question: is it possible to replace the values (1.7) by a single quantity $\nu(T)$ that is invariant against the choice of the fundamental decomposition and that measures T in some reasonable way?

As it turns out from the next investigations, $\sup_{p \in \Phi} p(T)$ cannot play the role of $\nu(T)$ since it becomes infinite for all operators T carrying some x_0 with $(x_0, x_0) = 0$ into Tx_0 with $(Tx_0, Tx_0) \neq 0$.

On the other hand, the definition

$$\nu(T) = \inf_{p \in \Phi} p(T) = \inf_{p \in \Phi} \sup_{\substack{x \in \mathfrak{H} \\ x \neq 0}} \frac{p(Tx)}{p(x)} \tag{1.8}$$

always provides a finite quantity.

Clearly, $\nu(T)$ given by (1.8) is not smaller than the τ-spectral radius of T. However, in order to accept $\nu(T)$ as a quantity reasonably measuring T, one ought to be sure that even for operators with τ-spectral radius zero $\nu(T)$ is not "too often" zero.

We have tried to verify the desired property by shifting the problem to another invariant quantity, denoted by $\nu_1(T)$ and obtained from $\nu(T)$ by interchanging the operations inf and sup:

$$\nu_1(T) = \sup_{\substack{x \in \mathfrak{H} \\ x \neq 0}} \inf_{p \in \Phi} \frac{p(Tx)}{p(x)}. \tag{1.9}$$

Since $\nu_1(T) \leqq \nu(T)$, it would be sufficient to prove that $\nu_1(T)$ is not too often zero.

Considering $\nu_1(T)$ instead of $\nu(T)$ has the practical advantage that taking the infimum over Φ, an unexplored type of operation, must now be carried out for a single pair x, Tx rather than all pairs simultaneously. In fact,

$$\nu_1(T) = \sup_{\substack{x \in \mathfrak{H} \\ x \neq 0}} q(x, Tx), \tag{1.10}$$

where

$$q(x, y) = \inf_{p \in \Phi} \frac{p(y)}{p(x)} \qquad (x, y \in \mathfrak{H};\ x \neq 0). \tag{1.11}$$