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ON A CONJECTURE OF FEJES TÓTH

by

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FEJES TÓTH [1] made the following conjecture. "If in a packing of translates of a square each square has at least six neighbours then the density of the packing is at least 11/15." Here a "square" is, for instance, the square $0 < x < 1$, $0 < y < 1$, a "packing" is a nonempty family of pairwise disjoint sets and, finally, two sets are said to be "neighbours" if their closures have a nonempty intersection. FEJES TÓTH has constructed a packing of density 11/15 which satisfies his requirements (see Fig. 1) and observed that every packing satisfying his requirements has density at least 2/3. It is not difficult to show [2] that in the investigation of this problem we can restrict ourselves to squares forming a grid, i.e., a set of squares joining along whole sides and filling the plane completely. For grids, HANANI improved the lower bound into 5/7 and restated the conjecture as follows. "In the planar square grid, color the squares blue and red, so that (a) there is at least one blue square and (b) each blue square has at least six blue neighbours. Then the density of the set of blue squares is at least 11/15." We shall prove this conjecture.

By the *order* of a square S we shall mean the number of the neighbours of S having the same color as S. By (b), there are no blue squares of order smaller than six. Moreover, there are no red squares of order greater than three. (This has been also observed by Hanani.) Let B_i (resp. R_i) be the set of all the blue (resp. red) squares of order i; let b_i (resp. r_i) be the density of B_i (resp. R_i). Then evidently

$$r_0 + r_1 + r_2 + r_3 + b_6 + b_7 + b_8 = 1.$$

Moreover, counting the red-blue connections we obtain

$$8r_0 + 7r_1 + 6r_2 + 5r_3 = 2b_6 + b_7.$$

Now, Hanani's bound follows since

$$\begin{aligned} 7(b_6 + b_7 + b_8) &\geq -3r_0 - 2r_1 - r_2 + 7b_6 + 6b_7 + 5b_8 = \\ &= 5(r_0 + r_1 + r_2 + r_3 + b_6 + b_7 + b_8) - \\ &- (8r_0 + 7r_1 + 6r_2 + 5r_3 - 2b_6 - b_7) = 5\,. \end{aligned}$$

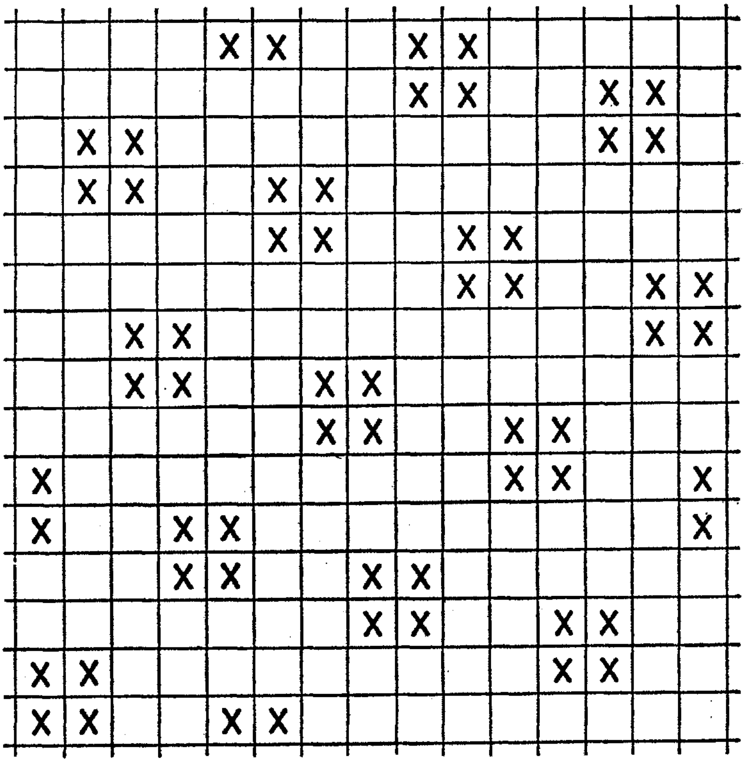

Fig. 1

To prove $b_6 + b_7 + b_8 \geq 11/15$, it will be enough to show that

$$(*) \qquad r_3 \leq 4r_0 + 2r_1 + 2b_7 + 4b_8,$$

for then

$$15(b_6 + b_7 + b_8) \geq -r_0 - r_1 - r_2 + 15b_6 + 15b_7 + 15b_8 =$$
$$= 11(r_0 + r_1 + r_2 + r_3 + b_6 + b_7 + b_8) -$$
$$- 2(8r_0 + 7r_1 + 6r_2 + 5r_3 - 2b_6 - b_7) +$$
$$+ (4r_0 + 2r_1 - r_3 + 2b_7 + 4b_8) \geq 11$$

as desired.

To prove (*), we first observe that the red squares of order three come in two by two quadruples. Let Q denote the set of these quadruples; set $q = r_3/4$. It will suffice to construct a bipartite graph G together with disjoint subsets S, T of R_1 such that